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On quantum state observability and measurement

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Abstract
We consider the problem of determining the state of a quantum system given one or more readings of the expectation value of an observable. The system is assumed to be a finite-dimensional quantum control system for which we can influence the dynamics by generating all the unitary evolutions in a Lie group. We investigate to what extent, by an appropriate sequence of evolutions and measurements, we can obtain information on the initial state of the system. We present a system theoretic viewpoint of this problem in that we study the *observability* of the system. In this context, we characterize the equivalence classes of indistinguishable states and propose algorithms for state identification.

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1. Introduction

Given a control system

$$\dot{x} = f(t, x, u) \tag{1}$$

where u represents the control function, x the state varying on a manifold M, with output $y = y(x)$, denote by $x(t, u, x_0)$ the solution of (1) with control u, initial condition x_0, at time t. Two states x_{01} and x_{02} are said to be *indistinguishable* (see, e.g., [26]) if, for every control u and every time t, we have $y(x(t, u, x_{01})) = y(x(t, u, x_{02}))$. A system is said to be *observable* if no two states in M are indistinguishable.

In this paper, we are interested in the observability properties of quantum control systems whose dynamics are described by the Liouville's equation for the density matrix ρ (see, e.g., [3]),

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\rho = [H(u(t)), \rho]. \tag{2}$$

We shall restrict ourselves to the finite-dimensional case where ρ is an $n \times n$ matrix. The *Hamiltonian* $H(u(t))$ is an $n \times n$ Hermitian matrix, in general, function of one or more control functions $u(t)$. We have assumed here and will assume in the rest of the paper that we are dealing with closed (noninteracting with the environment if not through the control functions and during the measurement process) quantum system. We assume that we perform a measurement of the mean value of an observable, represented by the Hermitian matrix S. In this case the output y takes the form

$$y = \mathrm{Tr}(S\rho). \tag{3}$$

Since $\mathrm{Tr}(\rho) \equiv 1$, it will be convenient to replace ρ with the traceless matrix $\rho - \frac{1}{n} I_{n\times n}$ and S with the traceless matrix $S - \frac{\mathrm{Tr}(S)}{n} I_{n\times n}$. This has the effect of 'shifting' the value of the output by a constant $\mathrm{Tr}(S)$ value which does not play any role in the indistinguishability considerations that will follow. Therefore we will set in the following $\mathrm{Tr}(\rho) = 0$ and $\mathrm{Tr}(S) = 0$. The solution of (2) varies as

$$\rho(t) = X(t)\rho(0)X^*(t) \tag{4}$$

with X solution of the Schrödinger equation,

$$\dot{X}(t) = -\mathrm{i}H(u(t))X \qquad X(0) = I_{n\times n}. \tag{5}$$

From known results in the theory of quantum control (see, e.g., [16, 19, 23], and see [9, 18] for the non-bilinear case), X can be driven to every value in the Lie group $\mathrm{e}^{\mathcal{L}}$ corresponding to the Lie algebra $\mathcal{L}$ generated by $\mathrm{span}_{u\in\mathcal{U}}\{\mathrm{i}H(u)\}$ where $\mathcal{U}$ denotes the set of possible values for the control. With initial condition $\rho(0)$, Hermitian and with trace zero, the density matrix ρ can attain all the values in the orbit

$$\mathcal{O} := \{X\rho(0)X^* | X \in \mathrm{e}^{\mathcal{L}}\}. \tag{6}$$

A study of the observability of control systems involves two main things. First, one would like to collect, in equivalence classes, initial states that cannot be distinguished by varying the control and measuring the output (see the next two sections for definitions in our case). Second, one would like to have methods to infer the equivalence class of the initial state from appropriate sequences of measurements and evolutions. We consider these problems for quantum control systems in this paper.

The question of determination of the state of a quantum system from measurements is at the heart of quantum mechanics and it was already discussed by Pauli in [22]. Several contributions have appeared in recent years and a discussion of the problem in general terms can be found in [6], where, like in the present paper, the problem of determination of the *initial* state (as opposed to the *current* state) was described. We present in this paper a treatment of this topic from a system theoretic view point. In this context, our study is closely related to other studies of the observability of nonlinear systems [15, 17, 21] (see also [8] for systems varying on the Lie groups). However we consider here a specific model for which we can obtain more complete results. Moreover a new element appears in the treatment of quantum systems, that is the transformation of the state as a result of each measurement. This can take different forms according to the type of measurement considered (see, e.g., [4, 7, 10]). We shall mainly consider the case of Von Neumann measurement [24, 27] and discuss extensions to other cases.

The paper is organized as follows. In section 2 we define and describe the set of states that cannot be distinguished in one measurement. Then, we generalize in section 3 to states that cannot be distinguished in multiple measurements. The determination of the state from one or more measurements is discussed in section 4. Conclusions are given in section 5.

2. Indistinguishability and observability with a single measurement

In the following, S is the traceless Hermitian matrix representing the observable and $\rho(t, u, \rho_0)$ is the solution of (2) at time t, with initial condition equal to ρ_0, and control u.

Definition 1. *Two states ρ_1 and ρ_2 are indistinguishable in one step if, for every control function(s) u and every t, we have*

$$\mathrm{Tr}(S\rho(t, u, \rho_1)) = \mathrm{Tr}(S\rho(t, u, \rho_2)). \tag{7}$$

The definition asserts that two states ρ_1 and ρ_2 are indistinguishable if there is no admissible experiment involving only one measurement which would give different results with initial states ρ_1 and ρ_2. It is clear that indistinguishability in one step is an equivalence relation. The set of possible values for the density matrix will be denoted by $\mathcal{R}$. It is a convex subset of the vector space of $n \times n$ Hermitian matrices (with zero trace), $\mathrm{i}su(n)$ and, in general, the vector space spanned by the elements of $\mathcal{R}$ is the same as $\mathrm{i}(su(n))$. The elements of $\mathcal{R}$ are parametrized by $n^2 - 1 = \dim su(n)$ parameters[1].

Definition 2. *The system is observable in one step if ρ_1 and $\rho_2 \in \mathcal{R}$ are indistinguishable in one step only when $\rho_1 = \rho_2$.*

Instrumental in the characterization of classes of indistinguishable states is the vector space of $n \times n$ skew-Hermitian matrices,

$$\mathcal{V} := \oplus_{k=0}^{\infty} ad_{\mathcal{L}}^{k} \mathrm{i}S. \tag{8}$$

Here $ad_{\mathcal{L}}^{k}\mathrm{i}S$ is the space obtained by taking k Lie brackets of $\mathrm{i}S$ with elements in the Lie algebra $\mathcal{L}$. We shall call $\mathcal{V}$, *observability space*. If $B_1, \ldots, B_m$ is a set of generators of the Lie algebra $\mathcal{L}$, it follows from an application of the Jacobi identity (see appendix A) that the observability space $\mathcal{V}$ is spanned by the matrices[2]

$$ad_{B_{j_1}}^{k_1} ad_{B_{j_2}}^{k_2} \cdots ad_{B_{j_r}}^{k_r} \mathrm{i}S \tag{9}$$

with $k_1, \ldots, k_r \geqslant 0$, and $\{j_1, \ldots, j_r\} \in \{1, \ldots, m\}$. $\mathcal{V}$ is the smallest subspace of $su(n)$ *stable* under $\mathcal{L}$ and containing $\mathrm{i}S$.[3] $\mathcal{V}$ might not be the Lie Algebra, however, it is always a subspace of the Lie Algebra generated by $\mathrm{i}S, B_1, \ldots, B_m$ and therefore a subspace of $su(n)$. Therefore its dimension is bounded by $\dim su(n) = n^2 - 1$. Note that $\mathcal{V}$ can be calculated with an algorithm that, at each step, calculates the matrices of 'depth' $d+1$ from the matrices of depth d, where the depth is the number of Lie brackets performed, namely $k_1 + k_2 + \cdots + k_r$ in (9). The algorithm starts with the matrix $\mathrm{i}S$, which has depth 0, and ends when the dimension reaches $n^2 - 1$ or there is no increment in the dimension. By finite dimensionality, there is always a finite $\bar{k}$ such that

$$\mathcal{V} = \oplus_{k=0}^{\bar{k}} ad_{\mathcal{L}}^{k} \mathrm{i}S. \tag{10}$$

We have the following result that relates the partition of the state space into classes of indistinguishable states with the properties of the observability space $\mathcal{V}$.

Theorem 1. *The following three conditions are equivalent*

1. *The states ρ_1 and ρ_2 are indistinguishable in one step.*

[1] In the presence of special symmetries, a parametrization with fewer parameters can be given (see [2] for an example).
[2] $ad_R^k T := [R, [R, \ldots [R, T]]]$, where the Lie bracket is taken k times.
[3] $[\mathcal{L}, \mathcal{V}] \subseteq \mathcal{V}$.

2. *For every* $X \in e^{\mathcal{L}}$,

$$\mathrm{Tr}(X^* S X \rho_1) = \mathrm{Tr}(X^* S X \rho_2). \tag{11}$$

3. *For every* $F \in \mathcal{V}$,

$$\mathrm{Tr}(F\rho_1) = \mathrm{Tr}(F\rho_2). \tag{12}$$

Proof. The equivalence between conditions 1 and 2 simply follows from the fact that the set of values obtainable for ρ starting from $\rho(0) = \rho_{1,2}$ is described in (6), and from elementary properties of the trace.

Now assume (11) holds and choose k matrices $R_1, \ldots, R_k$ (not necessarily all different) in $\mathcal{L}$. Then, for every $k-$ple of real numbers $t_1, \ldots, t_k$ we have

$$\mathrm{Tr}(\mathrm{e}^{-R_1 t_1} \cdots \mathrm{e}^{-R_k t_k} \mathrm{i}S\, \mathrm{e}^{R_k t_k} \cdots \mathrm{e}^{R_1 t_1} \rho_1) = \mathrm{Tr}(\mathrm{e}^{-R_1 t_1} \cdots \mathrm{e}^{-R_k t_k} \mathrm{i}S\, \mathrm{e}^{R_k t_k} \cdots \mathrm{e}^{R_1 t_1} \rho_2). \tag{13}$$

Calculating the derivative, $\frac{\partial^k}{\partial t_1 \partial t_2 \cdots \partial t_k}{}_{t_1=t_2=\cdots=t_k=0}$, of both sides we obtain

$$\mathrm{Tr}\big(ad_{R_1} ad_{R_2} \cdots ad_{R_k} \mathrm{i}S\rho_1\big) = \mathrm{Tr}\big(ad_{R_1} ad_{R_2} \cdots ad_{R_k} \mathrm{i}S\rho_2\big) \tag{14}$$

which proves condition 3, since k and R_j, $j = 1, \ldots, k$, are not specified. To prove that condition 3 implies condition 1, let $F_1, \ldots, F_s$ be a basis of $\mathcal{V}$ with $F_1 = \mathrm{i}S$. Then we have, using (2),

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{Tr}(F_j \rho(t, u, \rho_{1,2})) = \sum_{k=1}^{s} a_{j,k}(t)\, \mathrm{Tr}(F_k \rho(t, u, \rho_{1,2})) \tag{15}$$

for some (time-varying) coefficients $a_{j,k}(t)$ depending on the control u. Therefore we have that $\mathrm{Tr}(F_j\rho(t, u, \rho_1))$ and $\mathrm{Tr}(F_j\rho(t, u, \rho_2))$ satisfy the same (linear) system of differential equations and since the initial conditions are the same, then

$$\mathrm{Tr}(F_j\rho(t, u, \rho_1)) = \mathrm{Tr}(F_j\rho(t, u, \rho_2)) \qquad j = 1, \ldots, s. \tag{16}$$

In particular, we have,

$$\mathrm{Tr}(S\rho(t, u, \rho_1)) = \mathrm{Tr}(S\rho(t, u, \rho_2)). \tag{17}$$

Therefore the two states are indistinguishable. □

The inner product $\langle \cdot, \cdot \rangle$ in $su(n)$ is defined as $\langle A, B \rangle = \mathrm{Tr}(AB^*)$. Theorem 1 states that two matrices in $\mathcal{R}$ are indistinguishable if and only if they differ by an element in $\mathcal{V}^{\perp}$. Therefore we can state the following criterion of observability in one step which is a consequence of theorem 1.

Theorem 2. *System (2) is observable in one step if and only if one of the following equivalent conditions are verified*

$$1. \quad \mathrm{span}_{X \in \mathrm{e}^{\mathcal{L}}} X^* \mathrm{i}SX = su(n) \tag{18}$$

$$2. \quad \mathcal{V} = su(n). \tag{19}$$

Remark. The notion of observability is closely related to the notion of *informational completeness* of observables as treated for example in [14]. A set of observables $\mathcal{B}$ is called informationally complete if $\mathrm{Tr}(B\rho_1) = \mathrm{Tr}(B\rho_2)$ for every $B \in \mathcal{B}$ implies $\rho_1 = \rho_2$. From condition 2 of theorem 1 and the definition of observability, we can say that a system is observable if and only if the set of operators $\{X^* SX | X \in \mathrm{e}^{\mathcal{L}}\}$ is informationally complete.

2.1. Relation between controllability and observability in one step

If $\mathcal{L} = su(n)$, namely the system is *operator controllable* [1], and $S \neq 0$, then it is also observable. In fact, in this case, we have

$$\text{span}_{X \in \mathrm{e}^{\mathcal{L}}} X^* \mathrm{i}SX = \text{span}_{X \in SU(n)} X^* \mathrm{i}SX = su(n). \tag{20}$$

To verify this we can more easily verify condition (19). Since $\mathcal{V}$ is a nonzero ideal of $su(n)$ and $su(n)$ is a simple Lie algebra, $\mathcal{V}$ must be equal to $su(n)$. Therefore we have

Corollary 3. *Controllability along with $S \neq 0$ implies observability in one step.*

The converse of corollary 3 is not true not only because we may have the equality

$$\{X^*SX | X \in \mathrm{e}^{\mathcal{L}}\} = \{X^*SX | X \in SU(n)\} \tag{21}$$

even though $\mathcal{L} \neq su(n)$ [1, 25] but also because we may have (18) (19) even though (21) is not verified. A simple example of this can be found already in the $n = 2$ case by taking

$$\mathrm{i}S = \begin{pmatrix} \mathrm{i} & 1 \\ -1 & -\mathrm{i} \end{pmatrix} \qquad \mathcal{L} = \text{span}\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}. \tag{22}$$

2.2. First-order conditions for observability in one step

The case of the equality of the orbits in (21) is particularly favourable because we can give a different condition of observability which avoids the calculation of the repeated Lie brackets for $\mathcal{V}$ and involves only the calculation of Lie brackets of depth 1. We have the following proposition.

Proposition 4. *Assume $S \neq 0$. The system is observable in one step if one of the following two equivalent conditions is verified*

$$1. \quad \{X^*SX | X \in \mathrm{e}^{\mathcal{L}}\} = \{X^*SX | X \in SU(n)\} \tag{23}$$

$$2. \quad [\mathcal{L}, \mathrm{i}S] = [su(n), \mathrm{i}S]. \tag{24}$$

From a practical point of view condition (24) may be easier to verify since it involves calculation of the first-order Lie brackets only. The condition tells us that by calculating the first-order Lie brackets, we can infer the properties of $\mathcal{V}$ which is defined through higher order Lie brackets. If condition (24) is not verified we may still have observability.

Proof. The equivalence between the conditions (20) and (23) was proved in [1] although in a different context[4], therefore we shall not repeat the proof here. Clearly (23) implies (18) with (20) and (19) and therefore observability. □

Condition (24) can be verified by comparing the dimensions of the two vector spaces. The dimension of $[\mathrm{i}S, su(n)]$ can be expressed in terms of the multiplicity of the eigenvalues of iS (recall that iS is not zero and it has zero trace so it has at least two distinct eigenvalues). We have

$$\dim[\mathrm{i}S, su(n)] = 2 \sum_{j<k} n_j n_k \tag{25}$$

where n_j (n_k) is the multiplicity of the jth (kth) eigenvalue.

[4] There, S was the density matrix and we wanted to give practical conditions to verify that the set of possible density matrices that can be obtained by varying X in $\mathrm{e}^{\mathcal{L}}$ is the same as the largest possible one namely the one obtained by varying $X \in SU(n)$. This condition was called *density matrix controllability*.

If $\mathrm{i}S$ is known to be in a proper subspace $\mathcal{F}$ of $su(n)$ stable under $\mathcal{L}$ (e.g., $\mathcal{L}$ itself or $\mathcal{L}^{\perp}$) then we cannot have observability because $\mathcal{V} \subseteq \mathcal{F} \neq su(n)$.

Example. Two spin $\frac{1}{2}$ particles are interacting through Ising interaction and are driven by an electro-magnetic field in the x direction [11, 12]. The magnetic field couples with one of the spins only and we can detect the magnetization in the z direction. Denote by $\sigma_{x,y,z}$ the x, y, z Pauli matrices (see, e.g., [24])

$$\sigma_x := \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y := \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad \sigma_z := \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{26}$$

by $\mathbf{1}$ the 2×2 identity matrix and by $u = u(t)$ the x component of magnetic field. After appropriately scaling the parameters involved, the Hamiltonian H has the form

$$H = \sigma_z \otimes \sigma_z + u(t)\sigma_x \otimes \mathbf{1} \tag{27}$$

and the output matrix S is given by $S = \sigma_z \otimes \mathbf{1} + \mathbf{1} \otimes \sigma_z$. The dynamical Lie algebra $\mathcal{L}$ is spanned by $\mathrm{i}\sigma_z \otimes \sigma_z$, $\mathrm{i}\mathbf{1} \otimes \sigma_x$ and $\mathrm{i}\sigma_z \otimes \sigma_y$. We have from formula (25) $\dim[\mathrm{i}S, su(n)] = 8$ while $\dim[\mathrm{i}S, \mathcal{L}] = 2$. Therefore the sufficient criterion of observability of proposition 4 fails. Moreover since $\mathrm{i}S$ is in $\mathcal{L}^{\perp}$, and $\mathcal{L}^{\perp}$ is stable under $\mathcal{L}$, the system is not observable.

2.3. Decomposition of the state space

It is natural to decompose the state ρ as $\rho(t) = \rho_1(t) + \rho_2(t)$, with $\rho_1(t) \in \mathcal{V}$ and $\rho_2(t) \in \mathcal{V}^{\perp}$, for every t. Then we have

$$\dot{\rho}_1 = -\mathrm{i}[H(u), \rho_1] \tag{28}$$

$$\dot{\rho}_2 = -\mathrm{i}[H(u), \rho_2] \tag{29}$$

and

$$\mathrm{Tr}(S\rho(t)) = \mathrm{Tr}(S\rho_1(t)) \tag{30}$$

for every t. Therefore if we are interested in the effect on the output S we can parametrize only the component of ρ in $\mathcal{V}$.

3. Indistinguishability and observability with multiple measurements

We now generalize the above characterization of states that are indistinguishable after one measurement to states that are indistinguishable after k measurements, for general k. In fact it may happen that, even if two states give the same output function at the first measurement, for every control and at every time, they give different values at the second measurement. This is a consequence of the fact that the first measurement modifies the state. Modern quantum measurement theory (see, e.g., [4, 7, 10]) has studied ways to model the change in the state due to measurement as well as ways to integrate the measurement process in the framework of quantum dynamics. We shall remark at the end of this section on possible extensions to other types of measurements but will consider the simplest case where the quantum measurement postulate [24, 27] holds. This is also called Von Neumann (or Von Neumann–Lüders) measurement. More precisely, rewrite the observable matrix S (assumed nondegenerate) as

$$S = \sum_{j=1}^{n} \lambda_j a_j a_j^* := \sum_{j=1}^{n} \lambda_j \Pi_j \tag{31}$$

where a_j are the orthonormal eigenvectors of S, Π_j, $j = 1, \ldots, n$, are the associated projection matrices defined by $\Pi_j := a_j a_j^*$ and λ_j are the associated eigenvalues. Define the automorphism $\mathcal{P}$ in the space of (skew)Hermitian matrices

$$\mathcal{P}(F) = \sum_{j=1}^{n} \Pi_j F \Pi_j \tag{32}$$

which returns the diagonal part of F, if we are working in a basis where S is diagonal. If the state at the time of the measurement is $\rho(t, u, \rho_0)$, according to the measurement postulate, the state after the measurement is $\mathcal{P}(\rho(t, u, \rho_0))$. Assume that the experiment consists of an evolution for time t_1 with control u_1, followed by a measurement, followed by an evolution for time t_2 with control u_2, followed by a measurement, and so on, up to an evolution for time t_k with control u_k. The kth measurement, at time $t_1 + t_2 + \cdots + t_k$, gives the result

$$y_k(t_1, \ldots, t_k, u_1, \ldots, u_k, \rho_0) := \mathrm{Tr}(S\rho(t_k, u_k, \mathcal{P}(\rho(t_{k-1}, u_{k-1}, \mathcal{P}(\cdots\mathcal{P}(\rho(t_1, u_1, \rho_0))))))). \tag{33}$$

We can extend definitions 1 and 2 as follows:

Definition 3. *Two states ρ_1 and ρ_2 are indistinguishable in k steps if for every sequence of control function(s), $u_1, u_2, \ldots, u_k$, defined in intervals $[0, t_1)$, $[0, t_2), \ldots, [0, t_k]$, we have*

$$y_k(t_1, \ldots, t_k, u_1, \ldots, u_k, \rho_1) = y_k(t_1, \ldots, t_k, u_1, \ldots, u_k, \rho_2). \tag{34}$$

Definition 4. *A system is observable in k steps if no two states are indistinguishable in k steps.*

Definition 5. *Two states are indistinguishable if they are indistinguishable in k steps for every $k \geqslant 1$. A system is said to be observable if no two states are indistinguishable.*

It is convenient to rewrite the output at the kth measurement in terms of the values of the evolution operator X in (5) at the endpoints of the intervals $[0, t_1), \ldots, [0, t_k]$. We call these values of X, $X_1, \ldots, X_k$. Using (4), we have

$$y_k := y_k(X_1, \ldots, X_k, \rho_0) = \mathrm{Tr}(SX_k\mathcal{P}(X_{k-1}\mathcal{P}(\cdots\mathcal{P}(X_1\rho_0X_1^*)\cdots)X_{k-1}^*)X_k). \tag{35}$$

Therefore an alternative definition of indistinguishability in k steps can be given, that is ρ_1 and ρ_2 are indistinguishable if for every set of values $X_1, \ldots, X_k$ in $\mathrm{e}^{\mathcal{L}}$, $y_k(X_1, \ldots, X_k, \rho_1) = y_k(X_1, \ldots, X_k, \rho_2)$.

We can give conditions of indistinguishability and observability as in theorems 1 and 2, by introducing *generalized observability spaces*. More specifically, define the *observability space of order* 0, $\mathcal{V}_0 := \mathrm{span}\{\mathrm{i}S\}$, and the *observability space of order* 1, $\mathcal{V}_1 := \mathcal{V}$ in (8). The *observability space of order k*, $\mathcal{V}_k$, is defined recursively by

$$\mathcal{V}_k := \oplus_{j=0}^{\infty} ad_{\mathcal{L}}^{j}\mathcal{P}(\mathcal{V}_{k-1}). \tag{36}$$

It is the largest subspace of $su(n)$ containing $\mathcal{P}(\mathcal{V}_{k-1})$ and stable under $\mathcal{L}$. It also follows from a proof analogous to the one in appendix A that, if $B_1, \ldots, B_m$ is a set of generators of the Lie algebra $\mathcal{L}$, $\mathcal{V}_k$ is spanned by the matrices

$$ad_{B_{j_1}}^{k_1} ad_{B_{j_2}}^{k_2} \cdots ad_{B_{j_r}}^{k_r} F \tag{37}$$

with $F \in \mathcal{P}(\mathcal{V}_{k-1})$, $k_1, \ldots, k_r \geqslant 0$, and $\{j_2, \ldots, j_r\} \in \{1, \ldots, m\}$. Note also that it follows by induction, since $\mathcal{V}_0 \subseteq \mathcal{V}_1$, that

$$\mathcal{V}_{k-1} \subseteq \mathcal{V}_k \tag{38}$$

for every $k \geqslant 1$.

We have the following generalization of theorem 1.

Theorem 5. *The following three conditions are equivalent*

1. *The states ρ_1 and ρ_2 are indistinguishable in k steps.*
2. *For every $k-$ple $X_1, \ldots, X_k$ with values in $e^{\mathcal{L}}$,*

$$\mathrm{Tr}(X_1^* \mathcal{P}(X_2^* \mathcal{P}(\cdots \mathcal{P}(X_{k-1}^* \mathcal{P}(X_k^* S X_k) X_{k-1}) \cdots) X_2) X_1 \rho_1) \\ = \mathrm{Tr}(X_1^* \mathcal{P}(X_2^* \mathcal{P}(\cdots \mathcal{P}(X_{k-1}^* \mathcal{P}(X_k^* S X_k) X_{k-1}) \cdots) X_2) X_1 \rho_2). \tag{39}$$

3. *For every $F \in \mathcal{V}_k$,*

$$\mathrm{Tr}(F\rho_1) = \mathrm{Tr}(F\rho_2). \tag{40}$$

It follows from (40) and (38) that if two states are indistinguishable in k steps they are indistinguishable in r steps for every $r < k$. In other terms if we can distinguish two states in r steps we can distinguish them in $k > r$ steps as well.

Proof. If ρ_1 and ρ_2 are indistinguishable, then, for all the $X_1, \ldots, X_k$ in $e^{\mathcal{L}}$, we have $y_k(X_1, \ldots, X_k, \rho_1) = y_k(X_1, \ldots, X_k, \rho_2)$ in (35). Now note that, for a general ρ_0,

$$\begin{aligned} &\mathrm{Tr}(S X_k \mathcal{P}(X_{k-1} \mathcal{P}(\cdots \mathcal{P}(X_1 \rho_0 X_1^*) \cdots) X_{k-1}^*) X_k^*) \\ &\quad = \mathrm{Tr}(X_k^* S X_k \mathcal{P}(X_{k-1} \mathcal{P}(\cdots \mathcal{P}(X_1 \rho_0 X_1^*) \cdots) X_{k-1}^*)) \\ &\quad = \mathrm{Tr}(\mathcal{P}(X_k^* S X_k) X_{k-1} \mathcal{P}(\cdots \mathcal{P}(X_1 \rho_0 X_1^*) \cdots) X_{k-1}^*) \\ &\quad = \mathrm{Tr}(X_{k-1}^* \mathcal{P}(X_k^* S X_k) X_{k-1} \mathcal{P}(\cdots \mathcal{P}(X_1 \rho_0 X_1^*) \cdots)) \\ &\quad \cdot \\ &\quad \cdot \\ &\quad \cdot \\ &\quad = \mathrm{Tr}(X_1^* \mathcal{P}(X_2^* \mathcal{P}(\cdots \mathcal{P}(X_{k-1}^* \mathcal{P}(X_k^* S X_k) X_{k-1}) \cdots) X_2) X_1 \rho_0). \end{aligned} \tag{41}$$

Using this for $\rho_0 = \rho_1$ and $\rho_0 = \rho_2$ along with (35) we see that indistinguishability of ρ_1 and ρ_2 in k steps implies equation (39). The proof that condition 2 implies condition 3 is exactly analogous to the corresponding proof in theorem 1. The proof that condition 3 implies indistinguishability also is a generalization of the corresponding proof in theorem 1, with some more elements that we now illustrate. Consider a basis F_j, $j = 1, \ldots, s$, of $\mathcal{V}_k$ and derive a differential equation for $\mathrm{Tr}(F_j \rho(t, u_1, \rho_{1,2}))$. The differential equations corresponding to ρ_1 and ρ_2 are the same with the same initial conditions, because of the assumption (40). Therefore, in particular, at time t_1, we have

$$\mathrm{Tr}(F_j \rho(t_1, u_1, \rho_1)) = \mathrm{Tr}(F_j \rho(t_1, u_1, \rho_2)) \tag{42}$$

for every $F_j \in \mathcal{V}_k$ and therefore for every $F_j \in \mathcal{P}(\mathcal{V}_{k-1})$. If $\bar{F}_j$, $j = 1, \ldots, \bar{s}$, is a basis of $\mathcal{V}_{k-1}$, then we have

$$\mathrm{Tr}(\mathcal{P}(\bar{F}_j) \rho(t_1, u_1, \rho_1)) = \mathrm{Tr}(\mathcal{P}(\bar{F}_j) \rho(t_1, u_1, \rho_2)) \tag{43}$$

that is

$$\mathrm{Tr}(\bar{F}_j \mathcal{P}(\rho(t_1, u_1, \rho_1))) = \mathrm{Tr}(\bar{F}_j \mathcal{P}(\rho(t_1, u_1, \rho_2))). \tag{44}$$

Now, derive a differential equation for the variables $\mathrm{Tr}(\bar{F}_j \rho)$, with $\bar{F}_j$ a basis of $\mathcal{V}_{k-1}$, on the second interval of length t_2 and with control u_2. The function corresponding to ρ_1 satisfy the same differential equation as the function corresponding to ρ_2 and since the initial conditions are the same, from (44), we obtain that for every $\bar{F}_j$ in $\mathcal{V}_{k-1}$

$$\mathrm{Tr}(\bar{F}_j \rho(t_2, u_2, \mathcal{P}(\rho(t_1, u_1, \rho_1)))) = \mathrm{Tr}(\bar{F}_j \rho(t_2, u_2, \mathcal{P}(\rho(t_1, u_1, \rho_2)))). \tag{45}$$

This is, in particular, true for elements of $\mathcal{P}(\mathcal{V}_{k-2})$. Proceeding this way, after k steps, we obtain the equalities of outputs y_k in (33) for $\rho_0 = \rho_1$ and $\rho_0 = \rho_2$, for every k-tuple $t_1, \ldots, t_k$ and controls $u_1, \ldots, u_k$, and therefore indistinguishability. $\square$

An example of $\mathcal{V}_1 \neq \mathcal{V}_2$ is given by

$$S := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad \mathcal{L} := \operatorname{span}\left\{\begin{pmatrix} \mathrm{i} & 0 & 2 \\ 0 & -\mathrm{i} & 0 \\ -2 & 0 & 0 \end{pmatrix}\right\}. \tag{46}$$

We also have the following theorem concerning observability.

Theorem 6. *System (2) is observable in k steps if and only if one of the following equivalent conditions is verified*

1. $\operatorname{span}_{X_1, X_2, \ldots, X_k \in \mathrm{e}^{\mathcal{L}}} X_1^* \mathcal{P}(X_2^* \mathcal{P}(\cdots \mathcal{P}(X_{k-1}^* \mathcal{P}(X_k^* \mathrm{i} S X_k) X_{k-1}) \cdots) X_2) X_1 = su(n) \qquad (47)$

2. $\mathcal{V}_k = su(n). \qquad (48)$

A system is observable if and only if there exists a k such that one of the equivalent conditions (47), (48) is verified.

To check observability we only need to verify (48) for a finite number of k until we find a k such that $\mathcal{V}_{k-1} = \mathcal{V}_k$ or $\mathcal{V}_k = su(n)$.

It is obvious that since controllability ($\mathcal{L} = su(n)$) implies observability in one step it also implies observability in k steps for every k. The natural extension of the condition (23) of proposition 4 would be

$$\begin{aligned} &\{X_1^* \mathcal{P}(X_2^* \mathcal{P}(\cdots \mathcal{P}(X_{k-1}^* \mathcal{P}(X_k^* S X_k) X_{k-1}) \cdots) X_2) X_1 | X_1, X_2, \ldots, X_k \in \mathrm{e}^{\mathcal{L}}\} \\ &= \{X_1^* \mathcal{P}(X_2^* \mathcal{P}(\cdots \mathcal{P}(X_{k-1}^* \mathcal{P}(X_k^* S X_k) X_{k-1}) \cdots) X_2) X_1 | X_1, X_2, \ldots, X_k \in SU(n)\}. \end{aligned} \tag{49}$$

However we cannot give the Lie Algebraic condition for (49) (which would be an extension of (24) for this case). Note that (24) is essentially the equality of the tangent spaces at S of the two manifolds in (23). The main difficulty is that the two sets in (49) are not guaranteed to be manifolds. For example, if we consider

$$S := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{50}$$

and $\mathrm{e}^{\mathcal{L}} := SO(2)$, and $k = 2$, then we have

$$\{X_1^* \mathcal{P}(X_2^* S X_2) X_1 | X_1, X_2 \in \mathrm{e}^{\mathcal{L}}\} = \left\{ \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \middle| a, b \in \mathbb{R}, \sqrt{a^2 + b^2} \leqslant 1 \right\} \tag{51}$$

which is a manifold with boundary.

Like for the case of indistinguishability in 1 step, we can write

$$\rho(t) = \rho_1(t) + \rho_2(t) \tag{52}$$

with $\rho_1(t) \in \mathcal{V}_k$ and $\rho_2(t) \in \mathcal{V}_k^{\perp}$, which satisfy the equations (28)–(30). Therefore if we are interested in the effect on the output S we can parametrize only the component of ρ in $\mathcal{V}_k$. In particular, if $\mathcal{V}_k$ is the largest of the observability spaces we can neglect the component $\rho_2(t)$ of the state since it will not have any effect on any measurement.

Remark. The above treatment, which has been presented for Von Neumann measurements, can be extended to more general types of measurements (see, e.g., [4, 5, 7, 10, 13]). We have used the fact that, according to the measurement postulate, the state changes as $\rho \to \mathcal{P}(\rho)$.

For a more general measurement, with a countable set of possible outcomes $\mathcal{M}$, the state will change according to

$$\rho \to \mathcal{F}(\rho) := \sum_{m\in\mathcal{M}} \Phi_m(\rho). \tag{53}$$

The super-operators Φ_m are called *operations* and according to Kraus' representation theorem [20], under suitable assumptions, can be expressed as

$$\Phi_m(\rho) := \sum_k \Omega_{mk}\rho\Omega_{mk}^* \tag{54}$$

for a countable set of operators Ω_{mk}. Our treatment will go through by replacing $\mathcal{P}(\rho)$ with $\mathcal{F}(\rho)$. In particular, we can define a dual super-operator $\mathcal{F}^*$ acting on observables as

$$\mathcal{F}^*(S) := \sum_{m\in\mathcal{M}} \Phi_m^*(S) \qquad \Phi_m^*(S) := \sum_k \Omega_{mk}^* S\Omega_{mk}. \tag{55}$$

This has the property $\mathrm{Tr}(\mathcal{F}^*(S)\rho) = \mathrm{Tr}(S\mathcal{F}(\rho))$ and we can use this to extend the calculations in theorem 5. Moreover the definition of $\mathcal{V}_k$ in (36) has to be replaced by

$$\mathcal{V}_k := \oplus_{j=0}^{\infty} ad_{\mathcal{L}}^{j}\mathcal{F}^*(\mathcal{V}_{k-1}). \tag{56}$$

4. Initial state determination

We now investigate how much information on the initial state that we can extract from an experiment which alternates prescribed evolutions with measurements. We deal with a single experiment and with a single quantum system rather than with many copies of the same system, as it is done some times in this context. We shall assume, for simplicity, that the system is controllable namely $\mathcal{L} = su(n)$. Moreover, we can assume, without loss of generality, that the output matrix S is diagonal. We shall use the following formula (see (35), (39)) for the output at the kth measurement

$$y_k = \mathrm{Tr}(X_1\rho_0X_1^*\mathcal{P}(X_2^*\mathcal{P}(\cdots\mathcal{P}(X_{k-1}^*\mathcal{P}(X_k^*SX_k)X_{k-1})\cdots)X_2)) \tag{57}$$

for the unknown initial state ρ_0. Now, since every matrix of the type $\mathcal{P}(\cdot)$ is diagonal, it follows from (57) that it is only possible to obtain information on the diagonal elements of $X_1\rho_0X_1^*$ and therefore on at most $n-1$ independent parameters of the unknown matrix ρ_0. It is in fact possible to obtain *all* the $n-1$ independent diagonal elements of the matrix $X_1\rho_0X_1^* := \tilde{\rho}_0$. At the first measurement we obtain

$$y_1 = \mathrm{Tr}(\tilde{\rho}_0 S). \tag{58}$$

Then we choose X_2 as a permutation matrix so that $S_2 := X_2^*SX_2$ is still diagonal but the diagonal elements are a permutation of the diagonal elements of S. We also have $\mathcal{P}(S_2) = S_2$ so that, at the second measurement, we obtain

$$y_2 = \mathrm{Tr}(\tilde{\rho}_0 S_2). \tag{59}$$

Then we choose the evolution X_3 with $X_3 := \bar{X}_3X_2^*$ and $\bar{X}_3$ performing another permutation of the diagonal elements of S. $\bar{X}_3^*S\bar{X}_3 := S_3$. Therefore, the third measurement gives

$$y_3 = \mathrm{Tr}(\tilde{\rho}_0 S_3). \tag{60}$$

Continuing this way, we can obtain $n!$ equations for the diagonal elements of $\tilde{\rho}_0$, $x_1,\ldots,x_n$, i.e.

$$\sum_{k=1}^{n} a_{jk}x_k = y_j \qquad j = 1,\ldots,n! \tag{61}$$

where the elements a_{jk} are appropriate permutations of the diagonal elements of S. To this we have to add the equation[5]

$$\mathrm{Tr}(\tilde{\rho}_0) = \sum_{k=1}^{n} x_k = 0. \tag{62}$$

If S is not a scalar matrix, it is always possible to choose $n-1$ permutations and therefore $n-1$ equations in (61) that together with (62) have a unique solution. In fact the $n!+1 \times n$ matrix obtained by placing in the first $n!$ rows all the permutations of the diagonal elements of S and in the last row $1, 1, \ldots, 1$ has always rank n. The rank of this matrix is the same as the rank of a matrix obtained by adding to every row the last row $(1, 1, \ldots, 1)$ multiplied by an arbitrary constant. Therefore we can assume that the elements of the matrix are nonnegative and apply a lemma in appendix B.

As seen above, in the Von Neumann case, the number of independent parameters that can be inferred by a sequence of evolutions and measurements is bounded by the dimension of the system. This suggests to consider different types of measurements to obtain complete information on the initial state of the system. One possible scheme is as follows. Consider a system Σ_1 of dimension n, with unknown state ρ_1 and couple it with a (large) system Σ_2, of dimension m, whose state is known to be ρ_2. The density matrix of the coupled system ρ at time 0 is

$$\rho(0) = \rho_1 \otimes \rho_2. \tag{63}$$

This matrix has dimension nm and only n^2-1 parameters are not known. Now, if we let ρ evolve, after time t, the matrix $\rho(t)$ cannot in general be written as a tensor product, since the two systems are now *entangled* [28]. If we perform repeated Von Neumann measurement on the coupled system, we are able to obtain information on $nm-1$ independent parameters of ρ. Since ρ contains n^2-1 unknown parameters only, we may be able to obtain information on all of them if $m \geqslant n$. We give now a simple numerical example of this scheme.

The unknown state of a spin $\frac{1}{2}$ particle is represented by the density matrix (without shift of the trace)

$$\rho_1 = \begin{pmatrix} m & l \\ l^* & 1-m \end{pmatrix} \tag{64}$$

with m real. Two spin $\frac{1}{2}$ particles with known state

$$\rho_2 = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{2}{3} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{2}{3} \end{pmatrix} \tag{65}$$

are coupled with it. Therefore the unknown state

$$\rho_0 := \rho_1 \otimes \rho_2 \tag{66}$$

has only three unknown parameters. We can observe some linear combination of the spins in the direction. A possible associated matrix is given by

$$S = 4\sigma_z \otimes \mathbf{1} \otimes \mathbf{1} + 2\mathbf{1} \otimes \sigma_z \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \otimes \sigma_z \tag{67}$$

(see (26)) which is diagonal. From formula (57) and the previous discussion we can obtain the diagonal elements of the matrix $X_1 \rho_0 X_1^*$. Consider the vectors

$$e_1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad e_2 := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad v_1 := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad w_1 := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -\mathrm{i} \end{pmatrix}. \tag{68}$$

[5] Recall that, without loss of generality, we are considering density matrices with trace equal to zero rather than one.

If the first three columns of X_1^* are chosen as

$$\begin{aligned}\vec{x}_1 &:= e_1 \otimes e_1 \otimes e_1 \\ \vec{x}_2 &:= v_1 \otimes e_1 \otimes e_2 \\ \vec{x}_3 &:= w_1 \otimes e_2 \otimes e_1\end{aligned} \tag{69}$$

then we obtain for the diagonal elements

$$\begin{aligned}\vec{x}_1^* \rho_0 \vec{x}_1 &= \tfrac{1}{9} m \\ \vec{x}_2^* \rho_0 \vec{x}_2 &= \tfrac{1}{9}(1 + 2\,\mathrm{Re}(l)) \\ \vec{x}_3^* \rho_0 \vec{x}_3 &= \tfrac{1}{9}(1 + 2\,\mathrm{Im}(l)).\end{aligned} \tag{70}$$

From this we can extract the values of m and l.

5. Discussion and conclusion

In this paper, we have presented a treatment of the observability properties of quantum systems compatible with quantum measurement theory. We have focused on Von Neumann measurements but indicated extensions to more general types of measurements. We have given a characterization of states that cannot be distinguished in one or more measurements and conditions for observability. Contrary to most studies on observability of nonlinear systems (see, e.g., [26]) conditions for observability and indistinguishability are *global* in this case; however, observability does not always imply that it is possible to infer from appropriate evolutions and measurements all the parameters of the initial state. In fact, for Von Neumann measurements, there is a natural limit to the number of parameters of the state that can be derived. This does not improve if we consider measurements of different (and not necessarily commuting) observables. In this case, the result of the kth measurement still has the form (57) although now each projection $\mathcal{P}$ corresponds to a possibly different observable and S corresponds to the observable measured last. In this case the first $\mathcal{P}$ on the left is *always* the projection corresponding to the first measurement and therefore, once again, only at most $n - 1$ independent parameters of $X_1 \rho_0 X_1^*$ can be obtained. We have seen, in the previous section, that complete information on the initial state may be obtained by coupling the system with an auxiliary system whose state is known.

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Appendix A. Evaluation of $\mathcal{V}$ using a set of generators of $\mathcal{L}$

Let $B_1, \ldots, B_m$ a set of generators of $\mathcal{L}$. Denote by $\bar{\mathcal{V}}$ the space spanned by the matrices in (9). It is obvious that

$$\bar{\mathcal{V}} \subseteq \oplus_{k=0}^{\infty} ad_{\mathcal{L}}^k \mathrm{i}S. \tag{71}$$

To show

$$\oplus_{k=0}^{\infty} ad_{\mathcal{L}}^k \mathrm{i}S \subseteq \bar{\mathcal{V}} \tag{72}$$

we first show that

$$[\bar{\mathcal{V}}, \mathcal{L}] \subseteq \bar{\mathcal{V}}. \tag{73}$$

It is enough to show for elements F in a basis of $\mathcal{L}$ given by $B_1, \ldots, B_m$, and linearly independent (repeated) Lie brackets, $[F, \bar{\mathcal{V}}] \subseteq \bar{\mathcal{V}}$. We proceed by induction on the depth of F. If F is of depth 0, namely F is one of the matrices $B_1, \ldots, B_m$ then (73) follows from the definition of $\bar{\mathcal{V}}$. Now, let us assume (73) true for matrices F of depth $\leqslant d$ and let us show it for matrices F of depth $d + 1$. In particular, write F as $F := [Z, T]$, where Z is of depth d and T is of depth zero. If $\bar{V}$ is a matrix in $\mathcal{V}$, from the Jacobi identity, we obtain

$$[V, [Z, T]] = -[Z, [T, V]] - [T, [V, Z]] \tag{74}$$

since both terms on the right-hand side are in $\bar{\mathcal{V}}$, from the inductive assumption, we have that the term on the left-hand side is also in $\bar{\mathcal{V}}$, therefore we have proved (73). Now, from (73) we have

$$ad_{\mathcal{L}}\mathrm{i}S := [\mathrm{i}S, \mathcal{L}] \subseteq \bar{\mathcal{V}} \tag{75}$$

and from this

$$ad_{\mathcal{L}}^2\mathrm{i}S := [[\mathrm{i}S, \mathcal{L}], \mathcal{L}] \subseteq [\bar{\mathcal{V}}, \mathcal{L}] \subseteq \bar{\mathcal{V}} \tag{76}$$

where we have used (73). Proceeding this way, we see that for every $k \geqslant 0$

$$ad_{\mathcal{L}}^k\mathrm{i}S \subseteq \bar{\mathcal{V}} \tag{77}$$

which proves (72).

Appendix B

Lemma. *Let $x_1, \ldots, x_n$ be n non-negative numbers not all equal. Consider the matrix $A(x_1, \ldots, x_n) \in \mathbb{R}^{n!\times n}$ whose rows are the permutations of $x_1, \ldots, x_n$. Then the matrix $A = A(x_1, \ldots, x_n)$ has rank n.*

Proof. Let $2 \leqslant r \leqslant n$ be the number of different values assumed by the $\{x_i\}_{i=1,\ldots,n}$. Denote by $0 \leqslant d_1 < \cdots < d_r$ these values, and let l_i, for $i = 1, \ldots, r$ be the cardinality of $\{j | x_j = d_i\}$. Thus $\sum_{i=1}^r l_i = n$. We will prove our statement on induction on $r \geqslant 2$.

Case $r = 2$**.** We prove this part by induction on $n \geqslant 2$. If $n = 2$, then the statement is easily proved by computing the determinant of A. Let $n > 2$. Since all the columns of A sum up to the same value, which is strictly positive, setting

$$A' = \begin{pmatrix} 1, \ldots, 1 \\ A \end{pmatrix}. \tag{78}$$

We have that

$$\text{rank } A = \text{rank } A' = \text{rank} \begin{pmatrix} 1, \ldots, 1 \\ A \end{pmatrix} \tag{79}$$

where we have set.

Choose a value $x_{\bar{i}} \in \{x_1, \ldots, x_n\}$, such that $x_{\bar{i}} = d_1$, assuming that $l_1 \geqslant 2$ (otherwise choose it so that $x_{\bar{i}} = d_2$). Assume that we have rearranged the rows of A' in such a way that the first element of the second to the $(n-1)! + 1$-th row is $x_{\bar{i}}$. Note that this can be done since the rank remains unchanged. Then for $i = 2, \ldots, (n-1)! + 1$ we subtract from the i-row of A' the first row multiplied by $x_{\bar{i}}$. Note that if $d_1 = 0$ we leave the matrix unchanged. After this operation, the matrix A' has the following form:

$$A' = \begin{pmatrix} 1 & 1, \ldots, 1 \\ 0 & \\ \vdots & \tilde{A} \\ 0 & \\ & B \end{pmatrix}. \tag{80}$$

Note that $\tilde{A}$ is an $(n-1)! \times (n-1)$-matrix with the same structure of A and values $y_j = x_j - x_{\bar{i}} \geqslant 0$, in particular the y_j are either 0 or $d_2 - d_1$. Thus, by inductive assumption we have that rank $\tilde{A} = n-1$, which, in turn, implies rank $A' = n$ as desired. Had we chosen $x_{\bar{i}} = d_2$, we would have had all the values $y_j \leqslant 0$ with the two possible values 0 and $d_1 - d_2$, then we would have changed the sign of $\tilde{A}$ (which does not affect the rank) and applied the inductive assumption.

Case $r > 2$**.** We assume that the result is true for $r-1$. The idea of the proof is similar to the $r = 2$ case. Assume again that we have chosen $x_{\bar{i}} \in \{x_1, \ldots, x_n\}$, such that $x_{\bar{i}} = d_1$ and we have performed to the matrix A' (defined in (78)) the same operation as in the previous case to put A' in the form (80).

Now, if we prove that rank $\tilde{A} = n-1$ then we get rank $A' = n$. As before, $\tilde{A}$ is an $(n-1)! \times (n-1)$-matrix which is the same structure as A with values $y_j = x_j - x_{\bar{i}} \geqslant 0$ for $j \neq \bar{i}$. If $l_1 = 1$, then we are done by the inductive assumption since the numbers y_j assume $r-1$ different nonnegative values. If $l_1 > 1$, then we perform the same procedure as before starting with $\tilde{A}$ instead of A. Note that $\tilde{A}$ has $n-1$ different numbers, and is such that $d_1 = 0$ and the cardinality of the $\{y_j = 0\}$ is $l_1 - 1$. Thus we need to repeat this procedure l_1 times and then we can conclude by induction. □

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